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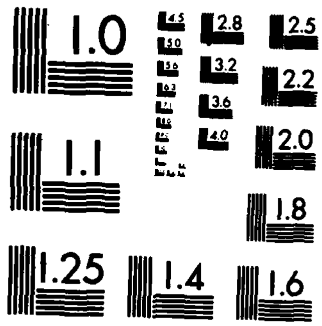
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AN ANALYTICAL MODEL OF PERIODIC WAVES IN SHALLOW WATER -- SUMMARY*

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ABSTRACT. An explicit, analytical model is presented of finite amplitude waves in shallow water. The waves in question have two independent spatial periods, in two independent horizontal directions. Both short-crested and long-crested waves are available from the model. Every wave pattern is an exact solution of the Kadomtsev-Petviashvili equation, and is based on a Riemann theta function of genus 2. These bi-periodic waves are direct generalizations of the well-known (simply periodic) cnoidal waves. Just as cnoidal waves are often used as one-dimensional models of "typical" nonlinear, periodic waves in shallow water, these bi-periodic waves may be considered to represent "typical" nonlinear, periodic waves in shallow water without the assumption of one-dimensionality.

EXTENDED SUMMARY. Waves in shallow water are familiar to virtually everyone. The objective of the work presented here is to construct an analytical model of waves in shallow water that is simple and explicit enough to be useful for engineering purposes, without being so simple that it fails to describe realistic wave patterns.

Some of the features that occur in "typical" waves in shallow water may be seen in Figures 1 and 2. Figure 1, taken from an ancient National Geographic (1933), shows a very regular train of one-dimensional waves off the coast of Panama. (By "one-dimensional", we mean that the surface pattern is one-dimensional, with virtually no variation along the wave crests. We call the waves in Figure 2 "two-dimensional", because the surface pattern is two-dimensional. In this terminology, there can be no three-dimensional waves on the two-dimensional water surface, even though the velocity fields may exhibit vertical structures. We emphasize that this is only a semantic convention.)

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Two features of the waves in Figure 1 are worth noting. The first is the very evident spatial period of the waves. The second is that the periodic wave pattern is far from sinusoidal: the wave crests are localized and rather steep, while the troughs are very broad and flat. The usual linear theory of infinitesimal water waves (e.g., Stoker, 1957) predicts a sinusoidal wave pattern, and the deviation of these waves from a sinusoidal shape is a measure of how nonlinear the waves are. Thus, Figure 1 suggests that a good model of "typical" waves in shallow water should admit waves that are: (i) periodic, and (ii) nonlinear.

Figure 2, taken by Mr. T. Toedtemeier off the coast of Oregon, shows an oblique interaction of two waves in shallow water. As in Figure 1, each wave has a sharp, localized crest and a broad, flat trough. Moreover, each of the interacting waves is part of a periodic wave train, but their wavelengths are so long, relative to the local water depth (about three feet, according to the photographer), that they behave like interacting solitary waves. The dominant effect of the interaction on each of the two wave crests is that each crest experiences a phase shift as a result of the interaction. The localized crests indicate that each (periodic) wave is individually nonlinear; the phase shift indicates that the interaction is also nonlinear.

The waves in Figure 1 happen to be one-dimensional, but those in Figure 2 clearly are not. In general, one would expect "typical" waves on the two-dimensional water surface to be two-dimensional.

Thus, we seek a model of waves in shallow water that are: (i) periodic; (ii) nonlinear; and (iii) two-dimensional. In fact, we want the simplest model possible that has these three properties. The main conclusion of this paper is that such a model now exists. In this summary we exhibit some consequences of the model. Full details will be published elsewhere (Segur & Finkel, 1984).

The Kadomtsev-Petviashvili (KP; 1970) equation,

$$\left(u_t + 6uu_x + u_{xxx}\right)_x + 3u_{yy} = 0, \tag{1}$$

is a scaled, dimensionless equation that describes the evolution of long water waves of moderate amplitude as they propagate primarily in one direction in shallow water of uniform depth, without dissipation. Mathematically, the KP equation generalizes the Korteweg-deVries (KdV) equation,

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2}$$

Physically, the derivation of (1) is very similar to that of (2), except that the waves in (2) are required to be strictly one-dimensional, while those in (1) may be weakly two-dimensional (see Ablowitz & Segur, 1979 for the derivation).

The KP equation also generalizes the KdV equation in the sense that both are completely integrable. In particular, Satsuma (1976) showed that the KP equation admits a two-soliton solution of the form,

$$u\ (x,y,t) = 2\partial_x^2 \ln F, \tag{3a}$$

where

$$F = 1 + \exp\ (\eta_1) + \exp\ (\eta_2) + \exp\ (\eta_1 + \eta_2 + A)\ , \tag{3b}$$

$$\eta_j = \kappa_j \left(x + \rho_j y - c_j t\right), \qquad c_j = \kappa_j + 3\rho_j^2\ , \tag{3c}$$

$$\exp\ (A) = \frac{\left(\kappa_1 - \kappa_2\right)^2 - \left(\rho_1 - \rho_2\right)^2}{\left(\kappa_1 + \kappa_2\right)^2 + \left(\rho_1 - \rho_2\right)^2}\ . \tag{3d}$$

A particular two-soliton solution is shown in Figure 3. The qualitative agreement between the wave patterns in Figures 2 and 3 is clear. Whether there is also quantitative agreement requires more detailed information about the ocean wave than is available.

This 2-soliton solution of the KP equation is nonlinear and two-dimensional, but it is not periodic. To achieve the desired model, we must generalize the 2-soliton solutions of the KP equation to include periodic waves.

Krichever (1976) showed that the KP equation admits periodic and quasi-periodic solutions in the form

$$u\ (x,y,t) = 2\partial_x^2 \ln \Theta \left(\phi_1, \ldots\ \phi_N\right) \tag{4}$$

where Θ is a Riemann theta function of genus N. In the simplest case, where $N = 1$,

$$\phi = \mu \left(x + \rho y - ct\right)\ , \tag{5}$$

$$\Theta(\phi) = \sum_n \exp\left(\frac{1}{2} bn^2 + i\ n\ \phi\right), \quad \mathrm{Re}\ (b) < 0\ ,$$

(4) yields the usual cnoidal wave solution that is familiar from KdV theory

(e.g., see Sarpkaya & Isaacson, 1981). Figure 4 shows a typical cnoidal wave solution of (1); it bears a clear resemblance to the wave shown in Figure 1. Moreover, cnoidal waves from KdV-theory have been validated in extensive experimental tests as accurate models of one-dimensional waves in shallow water.

Cnoidal waves are periodic and nonlinear, but they are not two-dimensional. To achieve the desired model, we must generalize the cnoidal wave solutions of the KP equation to two dimensions.

The required generalization comes by taking $N = 2$ in (4), and using certain results of Dubrovin (1981). This family of exact solutions are called KP solutions of genus 2; they have eight free parameters. There is a sense in which they represent an oblique and nonlinear superposition of two trains of cnoidal waves. In one limit, these solutions resemble a slightly modulated cnoidal wave, as in Figure 5a. In another limit, they become periodic generalizations of the 2-soliton solution, as in Figure 5b. However, they also represent waves outside of either limit, as in Figure 5c.

These figures demonstrate that it is possible to construct a variety of wave forms from the KP solutions of genus 2. By itself, this possibility does not qualify this family of solutions as a physical model of water waves. We must also give an explicit algorithm to specify every free parameter of the solution in terms of measured physical quantities. One such algorithm was given by Segur, Finkel & Philander (1983). A different algorithm, based on a different kind of data, is given by Segur & Finkel (1984). Once this algorithm is given, the model can be tested experimentally, and we invite interested experimentalists to do so.

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Figure Captions

1. Periodic waves in shallow water, from National Geographic, Vol. 63, No. 5, May 1933. Original caption read "As they near shallow water close to the coast of Panama, huge deep-sea waves, relics of a recent storm, are transformed into waves that have crests, but little or no troughs. A light breeze is blowing diagonally across the larger waves to produce a cross-chop. Three Army bombers, escorted by a training ship, are proceeding from Albrook Field, Canal Zone, to David, Panama."

2. Oblique interaction of two waves in shallow water. (Photograph courtesy of T. Toedtemeier).

3. Two soliton solutions of the KP equation. In (3), $\kappa_1 = \kappa_2 = 1$, $\rho_1 = 4 = -\rho_2$, $\exp(A) = 16/15$.

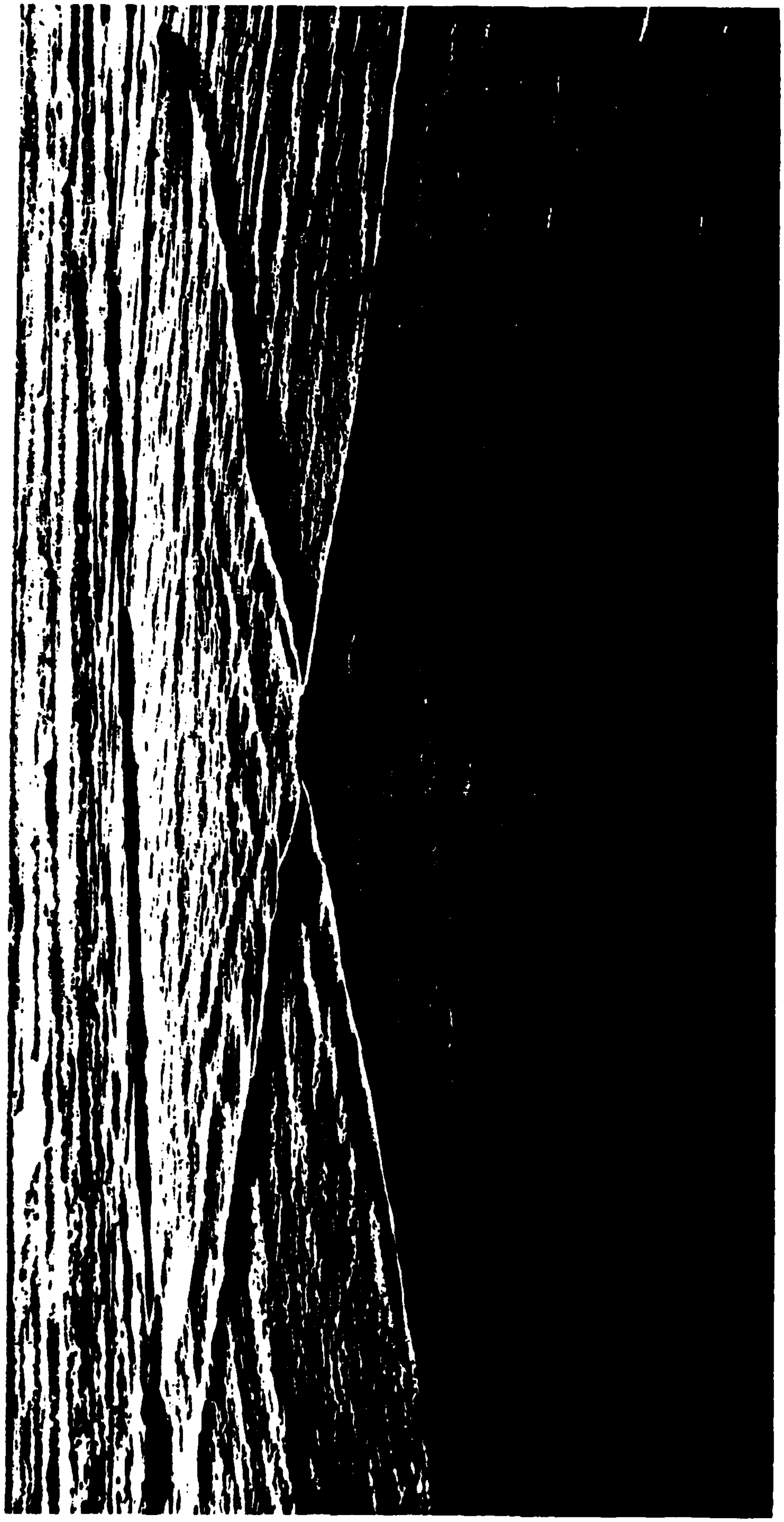
4. Cnoidal wave solution of the KP equation. In (5), $b = -3$, $\mu = 0.5$, $\rho = -0.43$.

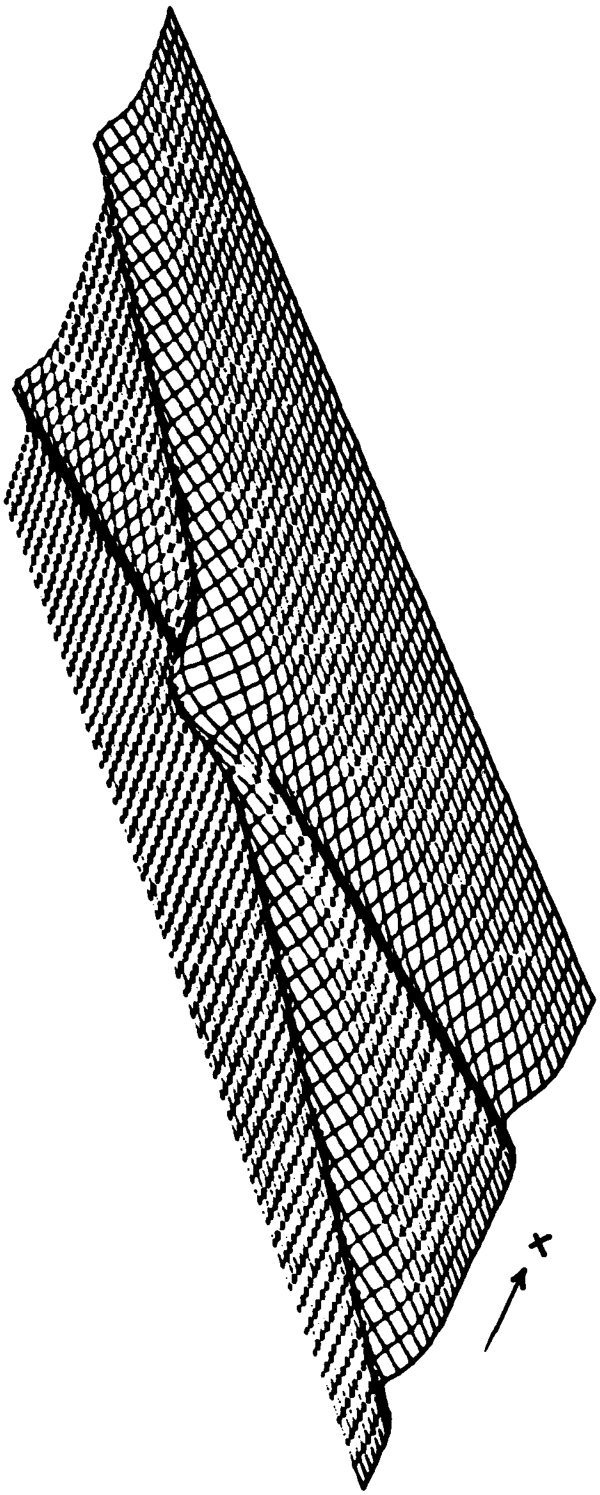
5. Some exact KP solutions of genus 2.
   a) One wave is dominant, so the weak waves simply modulates the strong one.
   b) Periodic generalization of 2-soliton solution.
   c) "typical" pattern of periodic waves of finite amplitude in shallow water.

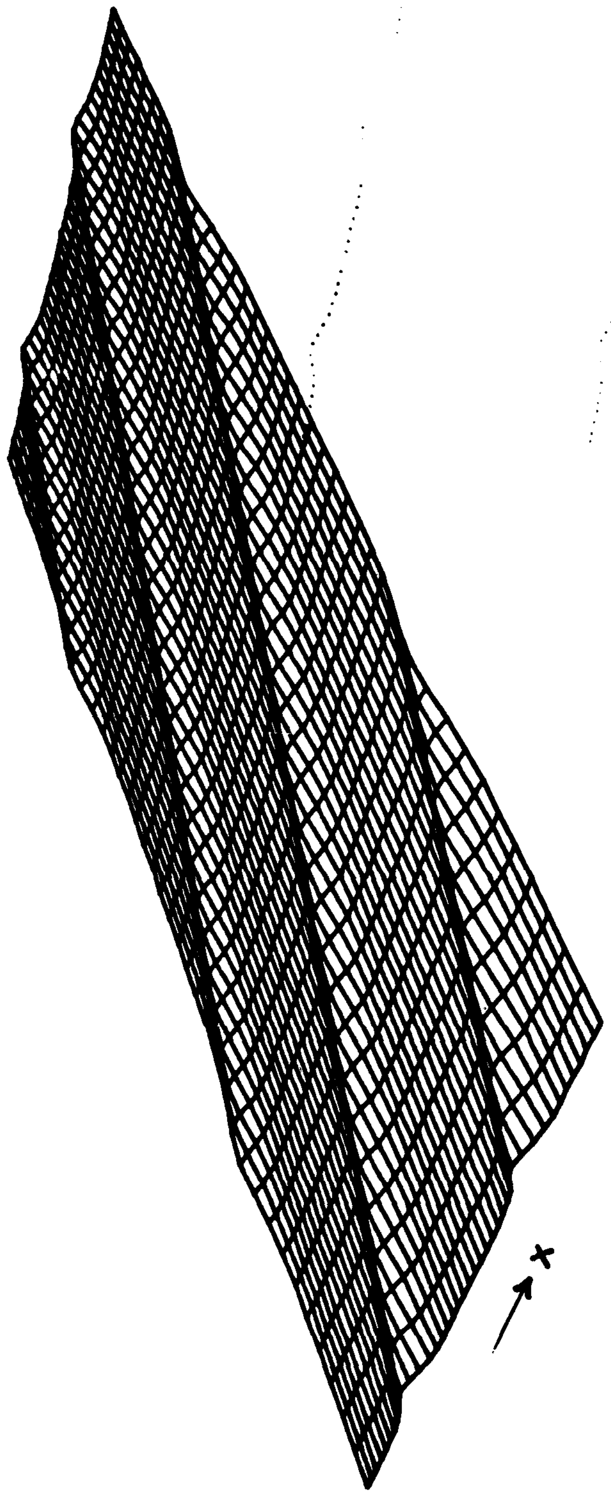
Official photograph, U. S. Army Air Corps

TINY FISHING BOATS RIDING A PACIFIC SWELL LOOK LIKE FLIES ON A CORRUGATED TIN ROOF

As they near shallow water close to the coast of Panama, huge deep-sea waves, relics of a recent storm, are transformed into waves that have crests, but little or no troughs. A light breeze is blowing diagonally across the larger waves to produce a cross-chop. Three Army bombers, escorted by a training ship, are proceeding from Albrook Field, Canal Zone, to David, Panama.



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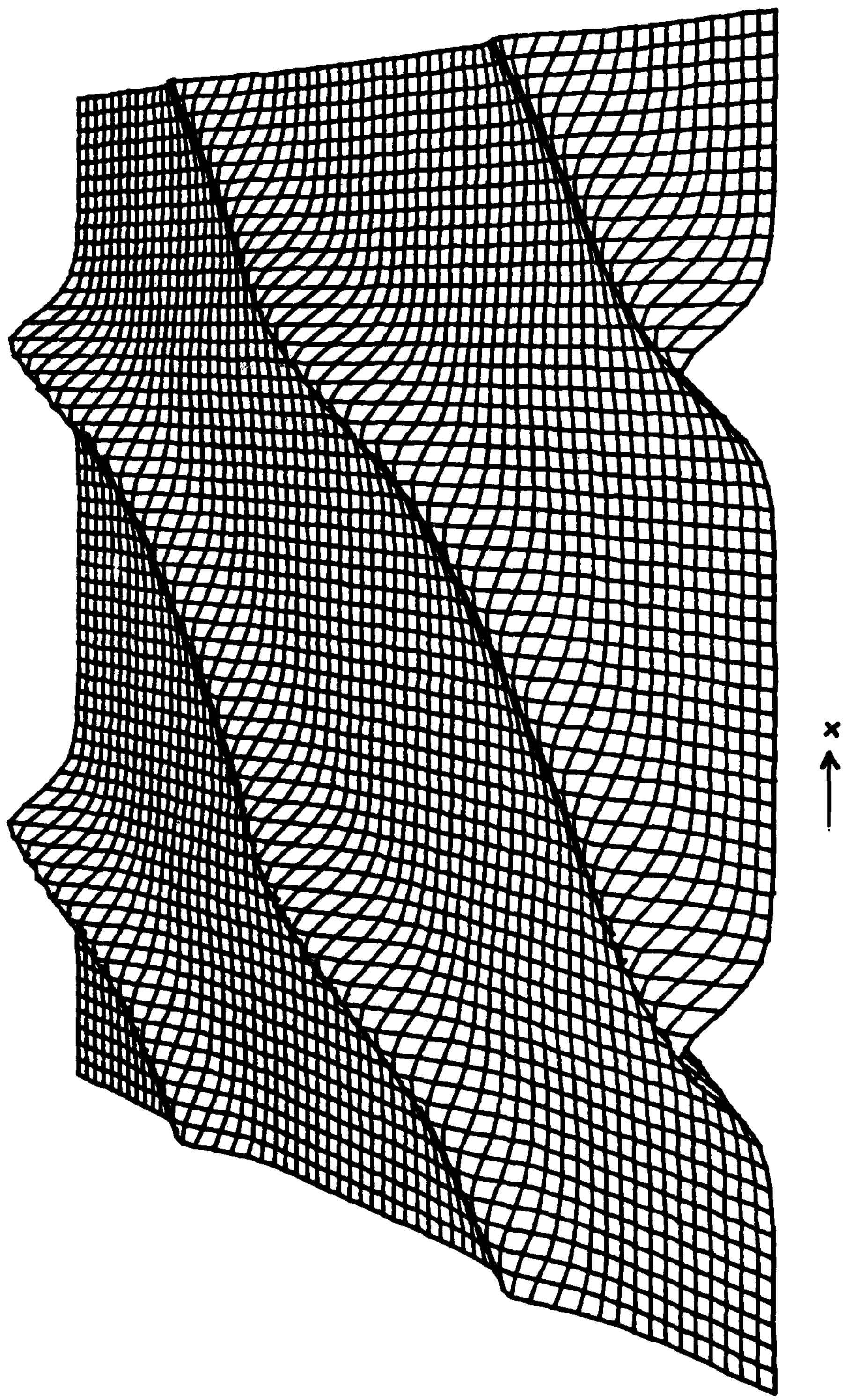
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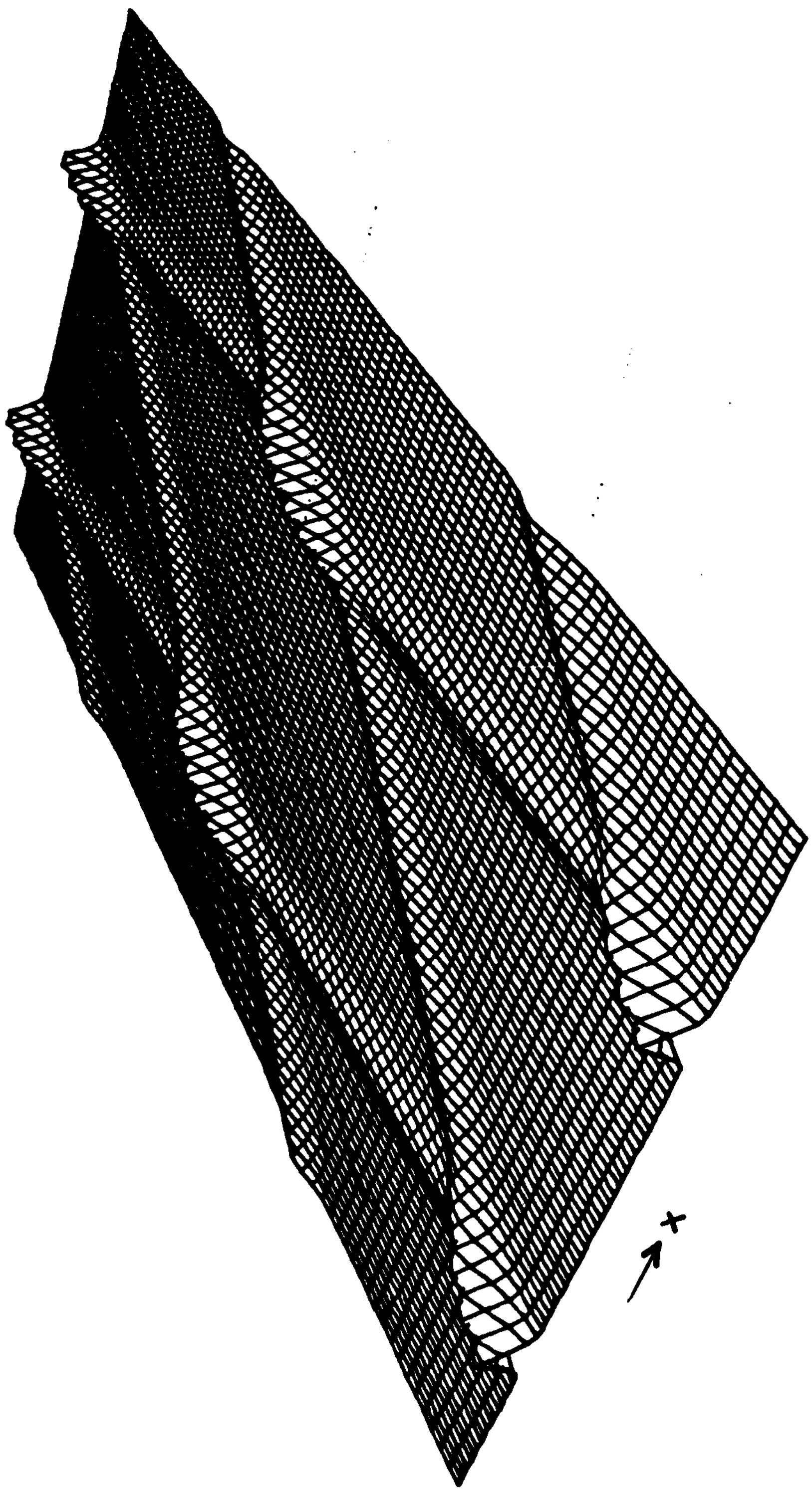
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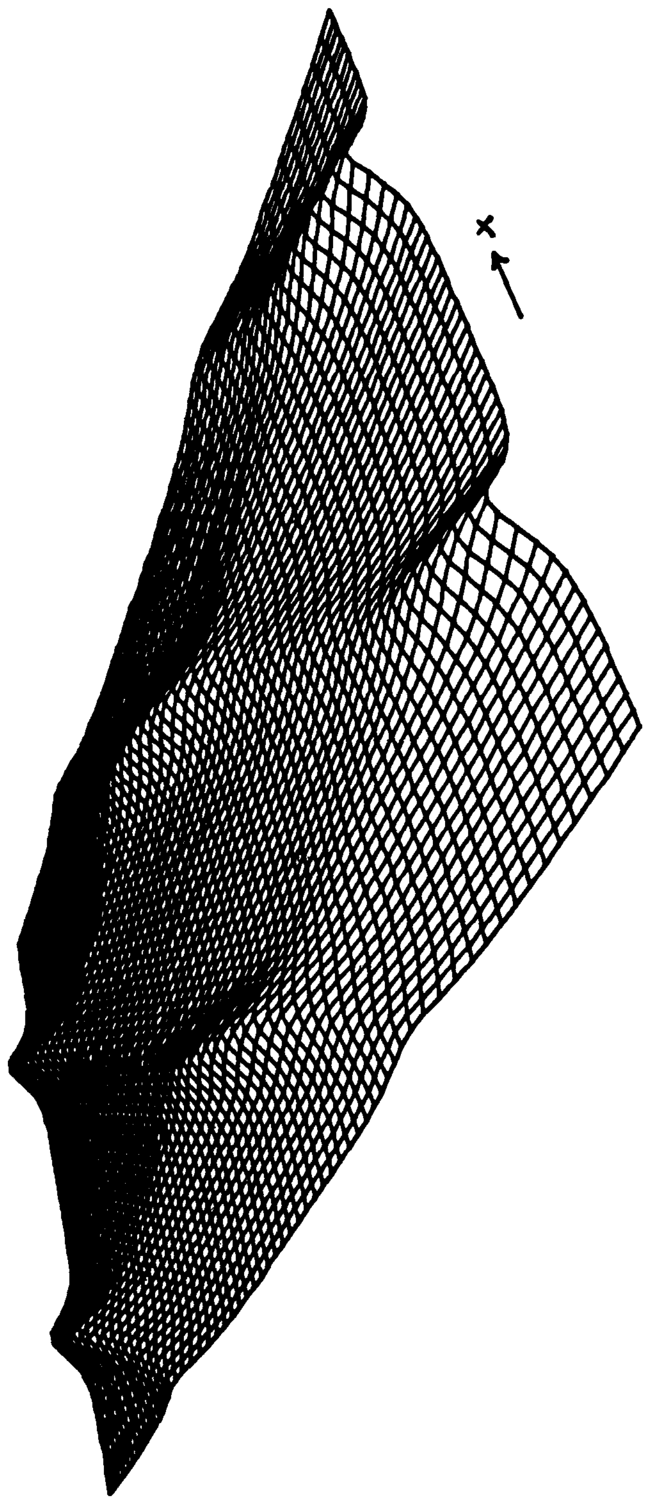

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